CALCULATION OF THE MOTION OF TWO-COMPONENT MEDIA

Yu. P. Zuikov and B. G. Kuznetsov

UDC 532.584

The one-dimensional motion of a viscous incompressible liquid in which particles are suspended is described by the mathematical model used in [1]. Two examples are discussed: the precipitation of particles from the suspension, and a boiling layer. The results are presented in the form of graphs.

1. Most of the results of investigations of the motion of two-component media are given in monographs [2-4]. Gol'dshtik [5] gives a qualitative analysis of the phenomena in a boiling layer and presents certain characteristic relations between the pressure and the particle density in the layer.

We consider the flow of a mixture of a viscous incompressible liquid and incompressible spherical particles of constant radius in a domain Q: $\{0 \le x \le L\}$ in the time interval $t \in [0, T]$. The acceleration due to gravity g is in the direction of the X axis. The equations describing the nonstationary one-dimensional motion of a two-component medium are [1]

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x}(\rho_i v_i) = 0, \quad \rho_i = m_i \alpha_i, \quad m_i = \text{const}, \quad \alpha_0 + \alpha_1 = 1$$

$$\rho_i\left(\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x}\right) = \rho_i g - \alpha_i \frac{\partial p}{\partial x} + 2\mu_0 \frac{\partial}{\partial x}\left(\alpha_i \frac{\partial v}{\partial x}\right) + (-1)^i \left\{k(v_1 - v_0) + \lambda_0\left[\beta \frac{\partial \alpha_1}{\partial x} + v \frac{\partial^2 \rho_1}{\partial x^2} - \frac{\partial^2}{\partial x^2}(\rho_1 v_1)\right]\right\} \qquad (i = 0, 1)$$

$$v = \frac{\rho_0 v_0 + \rho_1 v_1}{\rho_0 + \rho_1}, \quad k = \frac{\beta}{1 - k_1(\alpha_1/\alpha_1^*)^{2/3}}, \quad \beta = \frac{\varkappa \mu_0}{r^2} \qquad (1.1)$$

$$m_1 > m_0, \quad k_1 = \text{const}, \quad t \in [0, T], \quad x \in [0, L]$$

The subscripts 0 and 1 refer to the liquid and the particles respectively. We denote by p and v the pressure and the average mass velocity of the mixture; v_i, α_i, and ρ_i are respectively the velocities of the components, their specific volumes, and densities; the m_i are the true densities of the materials of the components; μ_0 and λ_0 are the viscosity and diffusion coefficients of the liquid; r is the radius of the particles; L is the length of domain Q; T is the characteristic time of the process. The shape factor $\varkappa = 4.5$ for spherical particles; α_1^* is the particle density for maximum close packing. For a tetrahedral arrangement of spherical particles a geometrical construction gives

$$\alpha_1^* = {}^2/_9\, \pi r^3 (n + 1)(n + 2), \quad n = \text{integer}\,(1.01965/r - 0.73205)$$

For $r \le 0.1$ mm, $\alpha_1^* \approx 0.7$.

The problem is to determine the functions p, v, v_i, α_i, and ρ_i which satisfy Eqs. (1.1) in the domain $Q_T = Q \times [0, T]$ and the boundary conditions

$$v_0(0, x) = U_0^\circ(x), \quad v_1(0, x) = U_1^\circ(x), \quad \alpha_1(0, x) = \alpha_1^\circ(x)$$
$$t = 0, \; x \in Q \qquad (1.2)$$

$$\left.\begin{array}{l} v_0(t, 0) = \varphi^\circ(t), \quad v_1(t, 0) = 0 \\ v_0(t, L) = \varphi^1(t), \quad v_1(t, L) = 0 \\ p(t, 0) = P_a = \text{const} \end{array}\right\}, \quad t \in [0, T] \qquad (1.3)$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 102-108, November-December, 1973. Original article submitted June 27, 1973.

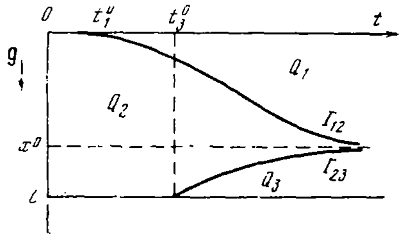

Fig. 1

The functions φ° and φ^1 must satisfy the condition

$$\alpha_0(t, 0)\,\varphi^\circ(t) = \alpha_0(t, L)\,\varphi^1(t)$$

We illustrate the main features of this problem by the example of the precipitation of the suspended particles.

2. The precipitation of particles from a suspension typically develops in the following way (Fig. 1). At time t_1° a zone of pure liquid Q_1 appears in the upper part of domain Q_T; at time t_3° a zone of dense precipitate Q_3 appears at the bottom of Q_T. Between them there is a zone Q_2 containing suspended particles. The zones are separated by the boundaries Γ_{12} and Γ_{23}. In the limit as $t \to T$ the boundaries Γ_{12} and Γ_{23} approach the straight line $x = x^\circ$ asymptotically. This state corresponds to the complete precipitation of the suspended particles.

For the problem of the precipitation of particles,conditions (1.2) and (1.3) are written in the form

$$v_0(0, x) = 0, \quad v_1(0, x) = 0, \quad \alpha_1(0, x) = \alpha_1^\circ(x), \quad t = 0, \quad x \in O \tag{2.1}$$

$$\left.\begin{array}{l} v_0(t, 0) = 0, \quad v_1(t, 0) = 0 \\ v_0(t, L) = 0, \quad v_1(t, L) = 0, \quad p(t, 0) = P_a \end{array}\right\}, \quad t \in [0, T] \tag{2.2}$$

A different set of equations holds in each zone. In zone Q_1 the equations of a single-phase viscous liquid apply, and these together with (2.1) and (2.2) give

$$\left.\begin{array}{l} \alpha_1(t, x) = 0, \quad \alpha_0(t, x) = 1 \\ v_0(t, x) = 0, \quad p(t, x) = P_a + m_0 g x \end{array}\right\}, \quad (t, x) \in Q_1 \tag{2.3}$$

In zone Q_2 system (1.1) must be solved; in Q_3

$$\left.\begin{array}{l} \alpha_0(t, x) = 1 - \alpha_1^* = \mathrm{const}, \quad \alpha_1(t, x) = \alpha_1^* \\ v_0(t, x) = 0, \quad v_1(t, x) = 0 \\ \dfrac{\partial p_0}{\partial x} = m_0 g, \quad \dfrac{\partial p}{\partial x} = m_1 g \end{array}\right\}, \quad (t, x) \in Q_3 \tag{2.4}$$

In zone Q_3 the particles are in contact with one another and the pressure between particles is transmitted directly through the liquid. As a result the pressure p in the mixture is different from the pressure p_0 in the liquid.

Equations (2.3), (1.1), and (2.4) form a closed system.

3. The laws of conservation of mass and momentum of the liquid and particles must hold on the boundaries Γ_{12} and Γ_{23}. These requirements determine the conditions the unknown functions must satisfy on these boundaries.

Let us consider a volume bounded by the points $x_-(t)$ and $x_+(t)$ moving so as to include the same set of liquid particles. Here $x_- \in Q_1$ and $x_+ \in Q_2$. Let $x_* \in \Gamma_{12}$, $x_- < x_* < x_+$. Then

$$\frac{dx_-}{dt} = 0, \quad \frac{dx_+}{dt} = v_{0+}(t, x_+), \quad \frac{dx_*}{dt} = v_{1+}(t, x_*) \tag{3.1}$$

where v_{0+} and v_{1+} are the velocities of the liquid and particles at the boundary Γ_{12} on the Q_2 side. Conservation of mass of the liquid is described by the equation

$$\frac{d}{dt}\left(\int_{x_-}^{x_*} m_0\,dx + \int_{x_*}^{x_+} \rho_0\,dx\right) = 0$$

Hence

$$m_0\left(\frac{dx_*}{dt} - \frac{dx_-}{dt}\right) + \rho_0(t, x_+)\frac{dx_+}{dt} - \rho_0(t, x_*)\frac{dx_*}{dt} + \int_{x_*}^{x_+} \frac{\partial \rho_0}{\partial t}\,dx = 0$$

or using (3.1) we obtain

$$\alpha_{0+}v_{0+} + \alpha_{1+}v_{1+} = 0 \tag{3.2}$$

In deriving (3.2) it was assumed that ρ_0 is a continuous function of x in zone Q_2 and the interval (x_*, x_+) is small.

Similar arguments for the particles and the mixture show that conservation of mass on Γ_{12} leads to the same condition (3.2).

Conservation of momentum of the liquid must be satisfied on Γ_{12}. Let (x_-, x_*) and (x_*, x_+) be two volumes moving with the velocity of the particles on the boundary Γ_{12}; $x_* \in \Gamma_{12}$. As $x_- \to x_+$ the law of conservation of momentum of the liquid in the first volume can be written in the form

$$\alpha_{0-}\left(-p_- + 2\mu_0\frac{\partial v_-}{\partial x}\right) - \rho_{0-}v_{0-}(v_{0-} - v_{1-}) = \alpha_{0-}\left[-p_+ + 2\mu_0\frac{\partial v_+}{\partial x}\right] - \rho_{0+}v_{0+}(v_{0+} - v_{1+})$$

The right-hand side of this equation contains the total momentum entering the first volume from the Q_2 side, and the square brackets contain the expression for the stress tensor of the mixture. Taking account of the fact that

$$\alpha_{0-} = 1, \quad \partial v_-/\partial x = 0, \quad \partial v_{0-}/\partial x = 0, \quad v_{0-} = 0$$

we obtain

$$-p_- = -p_+ + 2\mu_0\partial v_+/\partial x - \rho_{0+}v_{0+}(v_{0+} - v_{1+}) \tag{3.3}$$

The same arguments for the second volume give

$$-p_- = -p_+ + 2\mu_0\frac{\partial v_+}{\partial x} - \frac{\rho_{0+}v_{0+}}{\alpha_{0+}}(v_{0+} - v_{1+}), \quad \alpha_{0+} > 0 \tag{3.4}$$

In deriving (3.3) and (3.4) all body forces were assumed regular in the neighborhood of x_*.

Using (3.3), (3.4) and the law of conservation of momentum of the particles leads to the following relations on the line Γ_{12}:

$$v_{0-} = v_{0+} = 0, \quad \alpha_{0-} = \alpha_{0+} = 1, \quad p_+ = p_- + 2\mu_0\partial v_+/\partial x \tag{3.5}$$

At point x_* the conditions

$$\alpha_{1+}(t, x_*) = 0, \quad \partial\alpha_{1+}(t, x_*)/\partial x > 0 \tag{3.6}$$

must be satisfied.

The functions α_0 and v_0 must be continuous across the boundary Γ_{12}, but the pressure can experience a jump.

The laws of conservation of mass and momentum on the boundary Γ_{23} give the relations

$$\begin{gathered}(\alpha_{0-} - \alpha_0^*)\, v_* = \alpha_{0-}v_{0-}, \quad (\alpha_{1-} - \alpha_1^*)\, v_* = \alpha_{1-}v_{1-} \\ \rho_{0-}v_{0-}(v_* - v_{0-}) = \alpha_{0-}p_{0-} - 2\mu_0\alpha_{0-}\partial v_-/\partial x - \alpha_0^* p_{0+} \\ \rho_{1-}v_{1-}(v_* - v_{1-}) = \alpha_{1-}p_- - 2\mu_0\alpha_{1-}\partial v_-/\partial x - \alpha_1^* p_+\end{gathered} \tag{3.7}$$

The subscript minus denotes quantities on the Q_2 side, and the subscript plus refers to the Q_3 side. The quantity $v_* = dx_*/dt$ is the rate of change of the thickness of the precipitate layer, and $v_1 \neq v_*$. On Γ_{23} α_1 must satisfy the condition

$$\alpha_{1+} = \alpha_1^* = \text{const} \tag{3.8}$$

4. The accuracy of the solution of the problem depends on the correct selection of the boundaries of the media and satisfying conditions (3.5) and (3.7) on them.

If the domain Q_1 is known, the unknown functions within it are found in the form of the finite relations (2.3).

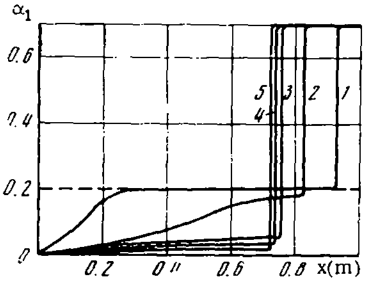

Fig. 2

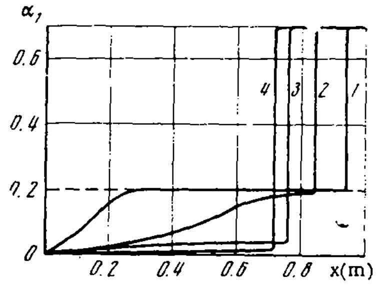

Fig. 3

Let us consider domain Q_2. Suppose the functions v^n, $v_0{}^n$, $v_1{}^n$, p^n, $\alpha_0{}^n$, $\alpha_1{}^n$, $\rho_0{}^n$, $\rho_1{}^n$ are known at time t^n in Q_2. Then the relation $dx_*/dt = v_{1*}$ can be used to find $x_*{}^{n+1}$ – the boundary Γ_{12} at the next instant t^{n+1}. The equation of continuity for the particles enables us to find $\alpha_1{}^{n+1}$ in Q_2 and consequently also the functions $\alpha_0{}^{n+1}$, $\rho_0{}^{n+1}$, and $\rho_1{}^{n+1}$. In addition the boundary Γ_{23} can be determined from Eqs. (3.7) and (3.8).

Using (2.2) and the equation of continuity gives the integral

$$\alpha_0 v_0 + \alpha_1 v_1 = 0 \tag{4.1}$$

After eliminating the pressure from the momentum equations and using (4.1) we obtain the equation for v_0

$$a\frac{\partial v_0}{\partial t} + b\frac{\partial^2 v_0}{\partial x^2} + c\frac{\partial v_0}{\partial x} + dv_0 = e \tag{4.2}$$

where the functions a, b, c, d, and e depend on t, x, v_0, α_1, $\partial\alpha_1/\partial x$, $\partial^2\alpha_1/\partial x^2$. On the curve Γ_{12} $\alpha_0 = 1$ and Eq. (4.2) contains a singularity $(\alpha_0 - 1)^{-1}$. But from (3.5) we have $v_0/\Gamma_{12} = 0$. At other points in Q_2 $\alpha_0 < 1$. Condition (3.7) enables us to find the value of v_0 on Γ_{23}, and then Eq. (4.2) can be solved in Q_2 numerically. We find $v_1{}^{n+1}$ from (4.1), v^{n+1} from the corresponding formula, and from any momentum equation with condition (3.5) we find the pressure p^{n+1}. Thus the unknown functions in Q_2 are found at time t^{n+1}.

Solving Eqs. (2.4) and (3.7) in zone Q_3 we obtain the unknown functions in the whole domain Q. They can be found at the next instant by repeating the procedure described.

The algorithm described was programmed in FORTRAN. The differential equations were integrated numerically by the finite difference method. The difference schemes used gave second-order approximations in the spatial variable and first-order in the time. The number of mesh points in the X direction was chosen equal to 100. The average time to calculate one variant of the problem on a BÉSM-6 computer did not exceed 10 min.

5. Various cases of the precipitation of particles were calculated as functions of m_0, m_1, μ_0, λ_0, r, and other parameters. The results are presented graphically for the precipitation of 0.1 mm radius particles having an initial density $\alpha_1{}^\circ = 0.2$. The particles were precipitated from water:

$$\mu_0 = 0.1_{10}^{-2}\ \text{kg/m·sec}, \quad \lambda_0 = 0.1_{10}^{-9}\ \text{m}^2/\text{sec},$$

$$m_0 = 1000\ \text{kg/m}^3$$

Figure 2 shows the particle density as a function of height in domain Q at time t = 60 sec. Curves 1, 2, 3, 4, and 5 correspond to particles with densities m_1 2, 5, 10, 15, and 20 times larger than the density of water m_0. The dashed line shows the initial density at t = 0.

Figure 3 shows the precipitation from water of glass spheres with r = 0.1 mm and $m_1 = 2\ m_0$. Curves 1, 2, 3, and 4 describe the behavior of α_1 at times t = 50, 200, 450, and 3600 sec.

Figure 4 shows the velocities of the particles (solid curves), the liquid (dash-dot), and the mixture (dashed) for the preceding example. The vertical straight lines 1, 2, and 3 correspond to the thickness of the precipitate layer at times t = 50, 200, and 450 sec. At t = 3600 sec all velocities are practically zero; i.e. the precipitation process has been completed. It is clear from this graph that the particles and the liquid have nearly the same speeds, but the fact that they are moving in opposite senses leads to additional forces impeding precipitation. Therefore according to the model of [1] the total duration of the precipitation process is substantially longer than in models which neglect the motion of the liquid [3].

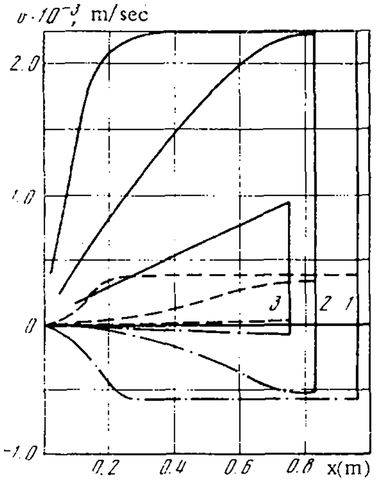

Fig. 4

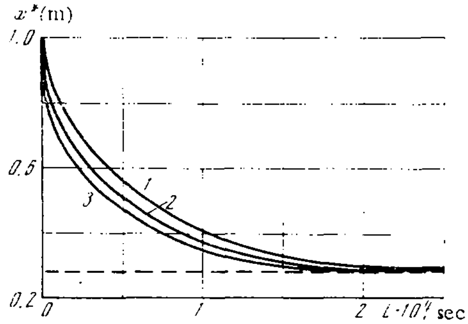

Fig. 5

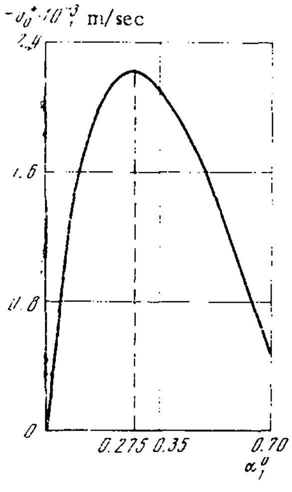

Fig. 6

Curves 1, 2, and 3 of Fig. 5 show the shapes of the Γ_{12} boundary for particles having material densities $m_1 = 2m_0$, $10m_0$, and $20m_0$. It is clear that the Γ_{12} curves for higher density particles approach the asymptote more rapidly; i.e. the precipitation process is completed more quickly.

6. Suppose a viscous incompressible liquid flows upward through a layer of solid spherical particles poured onto a horizontal mesh in a vertical cylindrical tube. The acceleration due to gravity is directed downward along the axis of the tube X. At a certain flow velocity of the liquid v_0^* the buoyant force on the particles becomes larger than the sum of the forces holding the layer of particles in a packed state (the weight of the particles, the interactions between particles, etc.). The layer begins to thicken, the distance between particles increases, and the particles become separated by layers of liquid. Since the passage of the liquid increases the cross section of the layer, the velocity of the liquid and the buoyant force decrease. A distribution of particle densities and velocities is established in the layer such that the buoyant force is balanced by the weight. The motion of the particles in the layer resembles the boiling process and by analogy such a layer is called a boiling layer.

A similar description of a boiling layer is given by Gol'dshtik [5]. There he presents an elementary theory of the layer based on empirical hypotheses and qualitatively describing the motion in a uniform boiling layer. A uniform boiling layer is one in which the flow of the liquid and the particles is distributed uniformly over the whole thickness of the layer. A uniform boiling layer has a clearly defined upper boundary separating it from the zone of "pure" liquid where the particle density is much lower than in the layer.

A uniform boiling layer is not always realized. Let us consider how the particle density α_1° in the layer and the velocity of the liquid v_0^* must be related for a boiling layer to be possible and stable. We solve the system of equations (1.1) for the conditions

$$v_0(0,x) = \begin{cases} v_0^*(1-\alpha_1^\circ), & x \in [0, l] \\ v_0^*, & x \in [l, L] \end{cases}, \quad \alpha_1(0,x) = \begin{cases} 0, & x \in [0, l] \\ \alpha_1^\circ, & x \in [l, L] \end{cases} \tag{6.1}$$

$$v_1(t,x) = 0, \quad x \in [0, L], \quad t = 0$$
$$\left.\begin{array}{ll} v_0(t,0) = v_0^*(1-\alpha_1^\circ), & v_1(t,0) = 0 \\ v_0(t,L) = v_0^*, & v_1(t,L) = 0 \end{array}\right\}, \quad t \in [0, T] \tag{6.2}$$

where $|L - l|$ is the initial thickness of the layer. By using (6.1) and (6.2) we solve (1.1) for v_0^* as a function of α_1°:

$$v_0^*(\alpha_1^\circ) = \frac{r^2}{\varkappa\mu_c}\left[1 - k_1\left(\frac{\alpha_1^\circ}{\alpha_1^*}\right)^{2/3}\right]\alpha_1^\circ(1-\alpha_1^\circ)(m_0 - m_1)g \tag{6.3}$$

This equation is plotted in Fig. 6. The figure shows that the solution of problem (1.1), (6.1), and (6.2) is not unique in the range $\alpha_1^\circ = 0$ to $\alpha_1^\circ = \alpha_1^* = 0.7$, the value for close packing. In order to select the stable solution test calculations of this problem were performed for Eq. (6.3). It was established that for a given velocity v_0^* a stable uniform layer is ensured only for values of $\alpha_1^\circ > 0.275$, the value for which $v_0^*(\alpha_1^\circ)$ is maximum. The result agrees qualitatively with the results in [5] where it is estimated that a stable boiling layer can exist for $\alpha_1^\circ > 0.35$.

Calculations were performed using $0.275 \le \alpha_1^\circ < \alpha_1^*$ and varying the speed of the liquid according to a definite law from zero to the value v_0^* previously found from (6.3). It turned out that a uniform stable boiling layer can exist if a dense precipitate cannot be formed in the bottom layer in the time required for the velocity of the liquid to reach the value v_0^*. Otherwise the velocity v_0^* turns out be insufficient to bring the layer into a uniform state. If v_0 is somewhat larger than v_0^* the upper part of the boiling layer is washed out, and its uniformity is destroyed. For $v_0 \gg v_0^*$ the particles float upward and a zone of pure liquid is formed at the bottom of the tube.

LITERATURE CITED

1. B. G. Kuznetsov, "On the equations of the hydrodynamics of multiphase systems," in: Numerical Methods of the Mechanics of Continuous Media [in Russian], Vol. 4, No. 1, Novosibirsk (1973).
2. S. Soo, Hydrodynamics of Multiphase Systems [Russian translation], Mir, Moscow (1971).
3. G. B. Wallis, One-Dimensional Two-Phase Flows, McGraw-Hill (1969).
4. M. É. Aérov and O. M. Todes, Hydraulic and Thermal Principles of the Operation of Devices with a Stationary and Boiling Granular Layer [in Russian], Khimiya, Leningrad (1968).
5. M. A. Gol'dshtik, "Elementary theory of the boiling layer," Prikl. Mekhan. i Tekh. Fiz., No. 6 (1972).